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Elliptically Symmetric Distributions: A Review and Bibliography

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Summary

This bibliography complete with discussion relates to the different areas in which elliptically symmetric distributions are considered. These areas range from inference to stochastic processes.

Key words: Spherical symmetry; Uncorrelated dependent samples.

1 Introduction

Elliptically symmetric distributions are appearing with increasing frequency in the literature. However, most authors seem to be unaware of the many publications in this area. This sometimes leads to a duplication of effort and results. This bibliography is to serve two purposes. First, to acquaint the general reader with the fact that there are alternatives to the Gaussian distribution which retain the structure of the Gaussian distribution, *i.e.* elliptical symmetry, while eliminating the specific form of the distribution. Secondly, to give the researcher interested in this area of statistics access to current and past publications.

An $(n \times 1)$ random vector $\mathbf{x}$ is said to be spherically symmetric if the distribution of $\mathbf{x}$ is identical to the distribution of $\mathbf{P}\mathbf{x}$ for all $(n \times n)$ orthogonal matrices $\mathbf{P}$. If we let $O(n)$ denote the group of $(n \times n)$ orthogonal matrices and $\mathcal{L}$ the distribution law, then we have that $\mathcal{L}(\mathbf{x}) = \mathcal{L}(\mathbf{P}\mathbf{x})$. An $(n \times m)$ random matrix $\mathbf{X}$ is left-spherically symmetric if $\mathcal{L}(\mathbf{X}) = \mathcal{L}(\mathbf{P}\mathbf{X})$, $\mathbf{P} \in O(n)$, right-spherically symmetric if $\mathcal{L}(\mathbf{X}) = \mathcal{L}(\mathbf{X}\mathbf{Q})$, $\mathbf{Q} \in O(m)$, and spherically symmetric if $\mathcal{L}(\mathbf{X}) = \mathcal{L}(\mathbf{P}\mathbf{X}\mathbf{Q})$. The transition from spherically symmetric to elliptically symmetric is made via an affine transformation. We now briefly discuss the papers included in the bibliography.

Three of the earliest papers pertaining to spherically symmetric distributions are those of Maxwell (1860), Bartlett (1934) and Hartman & Wintner (1940). They all give the following characterization of the Gaussian law. Let $\mathbf{x}' = (x_1, \dots, x_n)$ be a sample of n independently and identically distributed random variables. Then $\mathbf{x}$ has a spherically symmetric distribution if and only if $\mathbf{x}$ has a Gaussian law. More recently this characterization is noted by Kelker (1970), Thomas (1970), and Nash & Klamkin (1976).

One of the early modern papers is that of Lord (1954). He derives the characteristic function, $\phi_{\mathbf{x}}(\mathbf{t})$, when $\mathbf{x}$ has a spherically symmetric distribution. He also discusses the convolution of two independent spherically symmetric vectors and derives the probability density function (p.d.f.) of $(\mathbf{x}'\mathbf{x})^{1/2}$.

Kelker (1970) gives an extensive discussion of spherically symmetric distributions with p.d.f.'s of the form $f(\mathbf{x}) = \psi(\mathbf{x}'\mathbf{x})$. He also extends his discussion to include elliptically symmetric distributions with p.d.f.'s of the type $f(\mathbf{x}) = |\Sigma|^{-1/2} \psi\{(\mathbf{x} - \boldsymbol{\theta})' \Sigma^{-1} (\mathbf{x} - \boldsymbol{\theta})\}$. He gives a condition for the existence of a probability density function, derives marginal distributions, shows that the conditional means have the same linear structure for all elliptically symmetric distributions, and gives several characterizations of the Gaussian distribution. He also proves that, if $\boldsymbol{\theta} = \mathbf{0}$, $\Sigma = \mathbf{I}$ and the vector $\mathbf{x}$ is partitioned into $[\mathbf{x}'_1(1 \times n_1), \mathbf{x}'_2(1 \times n_2)]$, then the

distribution of $(\mathbf{x}'_1 \mathbf{x}_1 / \mathbf{x}'_2 \mathbf{x}_2)(n_2/n_1)$ has the usual central F distribution derived under Gaussian assumptions. This fact, that certain statistics have unchanged null distributions under a wider class of distributions, is also noted earlier by Bennett (1961) and Efron (1969).

Bennett (1961) examines the following distribution as an alternative to the Gaussian distribution in tests of significance in multivariate analysis. Let $\mathbf{x}(p \times 1)$ have density $f(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Lambda}) = K\{1 + n^{-1}Q(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Lambda})\}^{-\frac{1}{2}(n+p)}$, with $Q(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Lambda}) = (\mathbf{x} - \boldsymbol{\theta})' \boldsymbol{\Lambda}(\mathbf{x} - \boldsymbol{\theta})$. This is a multivariate generalization of Student's t distribution because, if $p = 1$, we get the univariate Student's t distribution with n degrees of freedom. When $\boldsymbol{\theta} = \mathbf{0}$ and $\boldsymbol{\Lambda} = \mathbf{I}$, Bennett notes that $\mathbf{x}' \mathbf{x} / (1 + \mathbf{x}' \mathbf{x})$ has a beta distribution with parameters $\frac{1}{2}p$ and $\frac{1}{2}n$.

Efron (1969) uses Fisher's (1925) essentially geometric derivation of Student's t distribution to show that this distribution remains unchanged if we assume an underlying spherically symmetric distribution rather than a Gaussian distribution.

This uniqueness of certain distributions under the weaker assumption of spherical symmetry is also noted by Ghosh & Pollack (1975) and Thomas (1970).

Ghosh & Pollack (1975) prove that if $x_1, \dots, x_n$ are random variables with unbounded support and with joint density spherically symmetric, then the joint distribution of $y_i = x_i/x_n$ ($1 \leq i \leq n-1$) is a multivariate Cauchy distribution.

Thomas (1970) looks at the univariate general linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ when $\boldsymbol{\varepsilon}$ has a spherically symmetric distribution. He shows that the usual t and F test statistics used for tests on $\boldsymbol{\beta}$ have unchanged null distributions for this wider class of spherically symmetric laws. He also gives expressions for the nonnull distributions for these two cases and for the statistics $\mathbf{x}' \mathbf{x}$. These distributions are not independent of the underlying distribution. Laurent (1974a), under less stringent conditions than Thomas, gives several other expressions for the noncentral distribution of $\mathbf{x}' \mathbf{x}$.

Thomas's (1970) unpublished work apparently is overlooked within the statistical community. A case in point is Zellner's (1976) consideration of the linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, for $\boldsymbol{\varepsilon}$ distributed as a multivariate t . This is a specific example of a spherically symmetric distribution. Zellner shows that $\hat{\boldsymbol{\beta}} = (\mathbf{X}' \mathbf{X})^{-1} \mathbf{X}' \mathbf{y}$ is the maximum likelihood estimate for $\boldsymbol{\beta}$ and, furthermore, that $\hat{\boldsymbol{\beta}}$ is a maximum likelihood estimate for $\boldsymbol{\beta}$ for all likelihood functions which are monotonically decreasing functions of $(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})$. He further adds that if second moments exist then $\hat{\boldsymbol{\beta}}$ is a minimum variance unbiased estimator. All this is noted by Thomas in his thesis. Zellner does not duplicate Thomas entirely, as he also considers problems of inference within a Bayesian framework.

Thomas, however, is not the first to use a spherically symmetric distribution instead of a Gaussian distribution in the analysis of linear models. His work was anticipated to some extent by Box (1952, 1953).

Box (1952) notes that the usual F statistics have the same null distribution for all $f(\mathbf{x}) = \psi(\mathbf{x}' \mathbf{x})$. Box (1953) makes the statement, in an abstract, that those tests which are uniformly most powerful under Gaussian assumptions are also uniformly most powerful for any spherical distribution in which ψ is a decreasing function. However, there is no published verification of these assertions, the validity of which accordingly remains open to question. Kariya & Eaton (1977) and Kariya (1977) also investigate power properties for several testing situations.

Kariya & Eaton (1977) assume that $f(\mathbf{x}) = \psi\{(\mathbf{x} - \boldsymbol{\mu}\mathbf{1})'(\mathbf{x} - \boldsymbol{\mu}\mathbf{1})\}$ for $\mathbf{1}' = (1, 1, \dots, 1)$ and that ψ is a nonincreasing function. The test for $H: \boldsymbol{\mu} = \mathbf{0}$ versus $K_1: \boldsymbol{\mu} > \mathbf{0}$ or $K_2: \boldsymbol{\mu} \neq \mathbf{0}$ is uniformly most powerful and the null distribution is the ordinary Student's t distribution. They also show that the test for $H: \boldsymbol{\mu} = \mathbf{0}$ versus $K: \boldsymbol{\mu} \neq \mathbf{0}$ is uniformly most powerful unbiased when ψ is also convex.

Kariya & Eaton (1977) and Kariya (1977) consider tests for serial correlation. Let the $(n \times 1)$ vector $\mathbf{x}$ have an elliptically symmetric distribution with $f(\mathbf{x}) = \psi[\mathbf{x}'\{\gamma \boldsymbol{\Sigma}(\lambda)\}^{-1} \mathbf{x}]$ ($\gamma > 0$) and $\boldsymbol{\Sigma}(\lambda)^{-1} = \mathbf{I}_n + \lambda \mathbf{A}$ for $\mathbf{A}$ a known $(n \times n)$ matrix. They test $H: \lambda = 0$ versus $K_1: \lambda > 0$ and

$K_2: \lambda \neq 0$. For these two tests the null distributions are identical to those derived under a Gaussian assumption. The one-sided test is uniformly most powerful if we assume that ψ is nonincreasing. The two-sided test is uniformly most powerful unbiased if we assume also that ψ is convex.

In terms of tests for scale parameters Chmielewski (1981) considers $H_0: \sigma_1^2 = \dots = \sigma_k^2$ and proves that the usual normal theory test statistics have both null and nonnull distributions which remain unchanged for the class of underlying spherically symmetric distributions.

So far we have cited research pertaining only to vector distributions. Much less has been done for matrices $\mathbf{X}(n \times m)$ with spherically symmetric distributions.

Hsu (1940) finds the p.d.f. of $\mathbf{S} = \mathbf{X}'\mathbf{X}$ when $\mathbf{X}(n \times m)$ has a p.d.f. of the form $f(\mathbf{X}) = \psi(\mathbf{X}'\mathbf{X})$. Ahmad (1972) investigates the multivariate general linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\theta} + \mathbf{E}$, with the assumption that $\mathbf{E}(n \times m)$ contains n independent and identically distributed spherically symmetric m vectors. He then finds the distribution of $\mathbf{E}'\mathbf{E}$. Unfortunately he seems to be unaware of Hsu's work. Furthermore, in his proof for the distribution of $\mathbf{E}'\mathbf{E}$ he assumes that $f(\mathbf{E}) = \psi(\mathbf{E}'\mathbf{E})$. Therefore, his proof reduces to a proof for the Wishart distribution because independence together with spherical symmetry imply that $f(\mathbf{E})$ is a Gaussian distribution.

Beran (1979) investigates tests for ellipsoidal symmetry under the assumption that each observation has an m -dimensional elliptically symmetric distribution. He shows that these goodness-of-fit tests have good asymptotic power over a broad spectrum of alternatives to elliptical symmetry.

Chmielewski (1980a, b) considers tests involving the scale matrix $\boldsymbol{\Sigma}$. Some cases considered are equality of k scale matrices, sphericity, block diagonal structure, and equicorrelatedness. In all cases the null distributions of certain invariant test statistics are exactly those found under normal theory. For tests of equality and sphericity the nonnull distributions are also those derived under normal theory. For the block diagonal structure, in the case of two blocks, a power property is proved for the underlying class of elliptically symmetric unimodal distributions.

Several papers are concerned with the structure and/or geometric properties of random matrices which are spherically symmetric. Bishop, Fraser & Ng (1979) show that a spherically symmetric distribution can be factored into independent components. These components have distributions which are uniform Gaussian, chi-squared, Wishart, matrix F , matrix t , matrix beta, or generalizations of these. Dawid (1978) gives characterizations of left-spherical, right-spherical and spherical distributions of random matrices which are submatrices of arbitrarily large arrays sharing the same orthogonal invariance. Ruben (1979) gives geometric representations of the m -parallelotope determined by m independent random points $\mathbf{x}_1, \dots, \mathbf{x}_m$, for each $\mathbf{x}_i \in R^n$ ($1 < m \leq n$) spherically symmetric.

Much of what we have discussed so far pertains to distribution theory and statistical inference. Spherically symmetric distributions, however, are employed in several other areas such as minimax estimation, stochastic processes, pattern recognition, fiducial inference, and probability inequalities. We first discuss each of these areas in turn and then discuss those papers which do not fit into any designated category.

2 Minimax estimation

Strawderman (1974) finds families of minimax estimators for the location parameters of a p variate ($p \geq 3$) distribution of the form

$$\int_0^\infty \frac{1}{(2\pi\sigma^2)^{p/2}} \exp\left\{-\frac{1}{2\sigma^2} (\mathbf{x} - \boldsymbol{\theta})' (\mathbf{x} - \boldsymbol{\theta})\right\} dG(\sigma),$$

where $G(\cdot)$ is a cumulative distribution function on $(0, \infty)$ and the loss is sum of squared errors.

Berger (1975) extends this by considering a density of the form

$$f(\mathbf{x}) = \int_0^\infty \frac{|\Sigma|^{-\frac{1}{2}}}{(2\pi)^{\frac{1}{2}p} \sigma^p} \exp\left\{-\frac{1}{2\sigma^2} (\mathbf{x} - \boldsymbol{\theta})' \Sigma^{-1} (\mathbf{x} - \boldsymbol{\theta})\right\} dG(\sigma),$$

and develops families of minimax estimators for $\boldsymbol{\theta}$ under a known quadratic loss.

Brandwein & Strawderman (1978) find families of minimax estimators for the location parameters of a p variate ($p \geq 3$) spherically-symmetric unimodal distribution with respect to general quadratic loss. They use the following definition of multivariate unimodality. A $(p \times 1)$ random vector $\mathbf{x}$ is said to have a spherically symmetric unimodal distribution about $\boldsymbol{\theta}$ if the p.d.f. of $\mathbf{x}$ with respect to Lebesgue measure is a nonincreasing function of $(\mathbf{x} - \boldsymbol{\theta})' (\mathbf{x} - \boldsymbol{\theta})$. Brandwein (1979) extends these results for $p \geq 4$ to distributions which are spherically symmetric and not necessarily unimodal.

3 Stochastic processes

Vershik (1964) defines a spherically invariant random process and shows that certain well-known results for the Gaussian process extend to this larger class; for example, closure under linear operators and the property of linear conditional expectations. McGraw & Wagner (1968) investigate bivariate elliptically-symmetric distributions. They use these distributions to describe the second-order moments of the transformation of a random signal by an instantaneous nonlinear device.

Blake & Thomas (1968) derive characteristic functions and discuss the relationship of spherically invariant processes to linear estimation theory and Gaussian processes. Picinbono (1970) compares the class of spherically invariant processes with a particular class of Gaussian compound processes.

Yao (1973) considers various basic statistical properties of multivariate characteristic functions and probability density functions associated with spherically invariant random processes. He then applies this to estimation and detection problems in communication theory.

Chu (1973) shows that elliptically symmetric distributions sometimes can be expressed as integrals of a set of Gaussian densities, that is

$$p(\mathbf{x}) = \int_0^\infty w(t) N_{\mathbf{x}}(t^{-1} \mathbf{C}) dt,$$

where $N_{\mathbf{x}}(t^{-1} \mathbf{C}) = (2\pi)^{-\frac{1}{2}n} \exp\{-\frac{1}{2}\mathbf{x}'(t^{-1} \mathbf{C})\mathbf{x}\}$. The function $w(t)$ is defined to be

$$(2\pi)^{\frac{1}{2}n} |\mathbf{C}|^{\frac{1}{2}} t^{-\frac{1}{2}n} L^{-1}\{f(s)\},$$

where $L(\cdot)$ is the Laplace transform operator, $L^{-1}(\cdot)$ is the inverse operator, and $f(s) = f(\frac{1}{2}\mathbf{x}' \mathbf{C}^{-1} \mathbf{x})$. Using this representation he derives marginal and conditional distributions. He then considers problems of computing optimal estimation, filtering, stochastic control, and team decisions in various linear systems.

Gualtierotti (1974) studies the equivalence properties of spherically invariant measures on a real and separable Hilbert space. He then uses this to give a likelihood ratio formula for equivalent spherically invariant measures (Gualtierotti, 1976).

Goldman (1976) finds the p.d.f. of $\mathbf{x} + \mathbf{n}$ when the random vector $\mathbf{x}$ has an elliptically symmetric distribution and $\mathbf{n}$ has a Gaussian distribution with $\mathbf{x}$ and $\mathbf{n}$ independent. He uses this to examine the problems of detecting a known signal vector in the presence of $\mathbf{x} + \mathbf{n}$, for $\mathbf{x}$ spherically symmetric.

Masreliez & Martin (1977) derive robust Bayesian estimates in the vector random effects linear model for two cases. In the first case the errors are Gaussian while the random vector

parameter is heavy-tailed nonnormal spherically symmetric. In the second case the distributional roles are reversed. These robust estimates are then used to obtain robust estimates for the Kalman filter model.

4 Pattern recognition

Cooper (1962) uses the hypersphere as a decision boundary in pattern recognition. He shows that this decision boundary is fully optimum for the class of spherically symmetric distributions. In particular he investigates the Gaussian, spherical Pearson type II, and spherical Pearson type VII laws.

Haralick (1977) considers pattern discrimination for elliptically symmetric vectors. A lower bound on the probability of correct classification is found in terms of either an incomplete beta or gamma integral for the case where the distributions have common scale matrices and a monotonic functional form.

5 Fiducial inference

All references cited here are for the multivariate structural model $\mathbf{Y}(n \times m) = \mathbf{X}\boldsymbol{\beta} + \mathbf{E}\boldsymbol{\Gamma}$. Fraser & Haq (1969) derive error and structural distributions for this model. As a special case, they assume $\mathbf{E}$ has a spherically symmetric distribution with $f'(\mathbf{E}) = \psi(\mathbf{E}'\mathbf{E})$.

Fraser & Haq (1970) derive the conditional distribution for the error given the response observations. Again, a special case considered is for $\mathbf{E}$ spherically symmetric.

Dawid (1977) considers inferences about $\boldsymbol{\beta}$, hypothesis testing and confidence interval estimation, under the assumption that $\mathbf{E}$ has a spherically symmetric distribution. Using studentization he gets a test statistic which has a matrix t distribution. The null distribution of this test statistic is identical to the one derived under a Gaussian assumption.

6 Probability inequalities

Das Gupta *et al.* (1971) discuss inequalities for probabilities of types

$$\int_{-\mathbf{h}}^{\mathbf{h}} |\boldsymbol{\Sigma}|^{-\frac{1}{2}} \psi(\mathbf{x}' \boldsymbol{\Sigma}^{-1} \mathbf{x}) d\mathbf{x}, \quad \int_{-\infty}^{\mathbf{l}} |\boldsymbol{\Sigma}|^{-\frac{1}{2}} \psi(\mathbf{x}' \boldsymbol{\Sigma}^{-1} \mathbf{x}) d\mathbf{x},$$

where $\mathbf{h}' = (h_1, \dots, h_n)$, $\mathbf{l}' = (l_1, \dots, l_n)$ are constant vectors; $h_i \geq 0$ ($i = 1, \dots, n$) and $\boldsymbol{\Sigma}$ is positive-definite. Their main result is the following. Let

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \sigma_{pp} \end{bmatrix}$$

be a $(p \times p)$ positive-definite matrix with $\boldsymbol{\Sigma}_{11}: (p-1) \times (p-1)$, and let $\mathbf{x}' = (x_1, \dots, x_p)$ be a random vector with density function $|\boldsymbol{\Sigma}|^{-\frac{1}{2}} \psi(\mathbf{x}' \boldsymbol{\Sigma}^{-1} \mathbf{x})$, where

$$\boldsymbol{\Sigma}_\lambda = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \lambda \boldsymbol{\Sigma}_{12} \\ \lambda \boldsymbol{\Sigma}_{21} & \sigma_{pp} \end{bmatrix} \quad (0 \leq \lambda \leq 1).$$

If C is a convex symmetric set in the Euclidean space $\mathbb{R}^{p-1}$, then

$$P_\lambda\{\mathbf{x}_1, \dots, \mathbf{x}_{p-1} \in C, |\mathbf{x}_p| \leq h\}$$

is nondecreasing in λ . Using this result they show that certain normal theory simultaneous confidence bounds extend to the class of elliptically symmetric distributions.

Wynn (1977) considers a spherically-symmetric bivariate probability distribution centered at zero. He gives an inequality comparing the probability content of two different polygons circumscribing the same fixed circle with center at zero. These results can be used to compare the widths of certain simultaneous confidence intervals.

7 Miscellaneous

Kingman (1963) considers the problem of random walks with spherical symmetry. Higgins (1975) shows that $f(\mathbf{x}) = \psi(\mathbf{x}'\mathbf{x})$ has a functional form which is uniquely determined by the distribution of $\mathbf{x}'\mathbf{x}$. Using this characterization he gives a geometrical method for constructing multivariate distributions which is based upon surface integral techniques.

Devlin, Gnanadesikan & Kettenring (1976) give a summary of distributional results associated with elliptically symmetric distributions. As the correlation matrices for all elliptically symmetric distributions having a given matrix Σ are identical, they discuss using a subclass of these distributions, the variance mixtures of Gaussian distributions, for simulating non-Gaussian alternatives in studies of measures of correlation.

Tashiro (1977) gives methods for generating uniform random points on the surface of an n -sphere. These methods can be used to simulate points which possess a particular spherically symmetric distribution.

Johnson & Ramberg (1977) actually derive computer algorithms for generating elliptically symmetric distributions. Convenient algorithms are given for generating bivariate elliptically symmetric variates.

Smith (1977) gives a nonparametric test for bivariate spherical symmetry using the empirical cumulative distribution function. He finds an asymptotic expansion for the null and nonnull distributions under simple alternatives.

Box & Hunter (1957) derive moments of a rotatable design of order d . These are also the moments up to order $2d$ of a spherical distribution.

Laurent (1973, 1975) discusses the concept of structure as related to spherically symmetric vector and matrix distributions.

Fraser (1953), in an abstract, notes that the usual rank tests remain valid if the errors in a linear model are assumed to have a spherically symmetric joint distribution.

Kingman (1972) considers an infinite sequence of random variables such that any finite sample, $x_1, \dots, x_n$, has a spherically symmetric joint distribution. He proves that there is a random variable V , real and nonnegative, such that conditional on V , the x_i are independent and normally distributed with mean zero and variance V .

Kudina (1975) looks at decompositions of spherically-symmetric vector distributions. In particular he generalizes the result that a distribution which is the projection of a uniform distribution in the unit sphere of $\mathbb{R}^{n+1}$ on $\mathbb{R}^n$ is decomposable.

Laurent (1974b) uses fractional calculus to attain projections and anti-projections of spherical distributions.

Maronna (1976) estimates the location vector and scale matrix by means of M -estimators when $\mathbf{x}_1, \dots, \mathbf{x}_n$ is a sample from a m -variate elliptically symmetric distribution.

Nimmo-Smith (1979) proves that if $\mathbf{y}(n \times 1)$ is a random vector with a multivariate distribution and finite second moments then all regressions are linear if and only if the distribution of $\mathbf{y}$ is, under some affine transformation, spherically symmetric. Kelker (1970) proved the sufficiency part of this result.

This concludes the discussion of the bibliography. To summarize we note that elliptically symmetric distributions have been employed in a wide range of settings for a wide range of purposes.

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Résumé

Cette bibliographie, discussion incluse, se rapporte aux différents secteurs dans lesquels les distributions symétriques elliptiques sont considérées. Ces secteurs couvrent d'inférence à les processus stochastiques.

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